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Gravitomagnetic and Gravitoelectric Waves in General Relativity: II

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Well-known linearizations of Einstein's field equations and isomorphisms with electromagnetic theory allow one to demonstrate, in principle, the existence of gravitomagnetic (GM) and gravitoelectric (GE) wavefields generated by transient nongravitational sources—as in the spin-up of a rigid sphere by an external torque (Tolstoy, I. (2001). *International Journal of Theoretical Physics* **40**(5), 1021–1031). Whereas such effects are currently too small to be measured in the laboratory, order of magnitude estimates suggest that major astrophysical events could generate signals (strains in the metric) observable by LIGO systems. GM/GE modes are entirely uncoupled from the quadrupole radiation of classic gravitational wave theory. However, both travel at light velocity c and, since quadrupole waves may be generated by, or in the neighborhood of, similar events, it is essential to demonstrate that LIGO array geometries can discriminate between them. This can be accomplished by determining arrival directions and polarization planes.

KEY WORDS: gravitomagnetism; gravitoelectric waves; gravitational radiation.

1. INTRODUCTION

The two classic linearization techniques applied to the field equations of general relativity—weak field approximations and post-Newtonian expansions in powers of v/c—will, under certain conditions, overlap and lead to equations similar to, or even isomorphic with, those of electromagnetism.

A case in point, examined in a previous paper (Tolstoy, 2001), is the prediction of gravitomagnetic (GM)/gravitoelectric (GE) modes of propagation as precise analogs of electromagnetic waves (for an em model consisting of charges of single sign interacting through purely attractive Coulomb forces). The case of a rigid sphere spun up from rest at $t = 0$ was used as an illustration, demonstrating the generation of a travelling GM/GE wave field that, for large t, yields the well-known steady state GM field (detection of which, for the spinning Earth, is the objective of the Stanford Gravity Probe B project—Everitt, 1988, 1992).

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This paper applies the standard conventions and notations used in Tolstoy (2001). Thus in an approximately Minkowski–Cartesian space the metric is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad \text{with} \quad \eta_{jk} = \delta_{jk}, \quad \eta_{00} = -1 \tag{1}$$

with coordinates x^α

$$\alpha, \beta = (0, 1, 2, 3), \quad \text{3-vector } \mathbf{x} = (x^j), \quad j, k = (1, 2, 3) \tag{2}$$

3-vector velocity	$\mathbf{v} = \mathbf{x}_{,t}$
0 (time) component	$v^0 = ic$
Mass density	ρ
Gravitational constant	$G = 6.67 \times 10^{-11}$ mksu
Speed of light	$c = 2.98 \times 10^8$ ms^{-1}
Isomorphic permeability	$\mu^* = 16\pi Gc^{-2} = 3.7 \times 10^{-26}$ mksu
3-momentum density ("mass current")	$\mathbf{j}^* = -\rho\mathbf{v}$
4-momentum density	$\boldsymbol{j}^* = (-ic\rho, -\rho v^k)$
Scalar potential	ϕ^*
3-vector potential	$\boldsymbol{\zeta}$
GM field	$\mathbf{B}^* = \nabla \times \boldsymbol{\zeta}$
GE field	$\mathbf{E}^* = -\boldsymbol{\zeta}_{,t} - \nabla\phi^*$
Strain	$h_{j0} = ic^{-1}\zeta_j$
Taking new scalar	$\Phi = 4\phi^*$
Define the 4-vector potential	$\mathbf{F}^* = (ic^{-1}\Phi, \boldsymbol{\zeta})$

The field equations are then

$$\Box^2 F^{*\alpha} = -\mu^\star j^{*\alpha} \tag{3}$$

$$F^{*\alpha}_{,\alpha} = 0 \tag{4}$$

These are gauge invariant (Tolstoy, 2001) and Eq. (4) is the Lorentz gauge. The system is formally identical (isomorphic) with em field equations for the em 4-potential.

The 4-vector $\mathbf{F}^{*\alpha}$ differs in general from the $h^{0\alpha}$ column of the weak field tensor $h^{\alpha\beta}$ of classic gravitational wave theory, which does *not* define a gauge invariant 4-vector.

It must also be kept in mind that the $\boldsymbol{j}^*(t)$ source terms in Eq. (3) will only generate radiative solutions $\mathbf{F}^*$ if the corresponding accelerations and velocities are of nongravitational origin (Tolstoy, 2001): in the present approximation systems of particles interacting solely via Coulomb forces cannot be sources of GM/GE radiation.

Taking $v^2/c^2 \ll 1$ and neglecting all quadratic terms, a classic result (Weinberg, 1972) also yields the force $\mathbf{f}$ on a unit mass particle with velocity $\mathbf{v}$:

$$\mathbf{f} = \mathbf{E}^* + \mathbf{v} \times \mathbf{B}^* \tag{5}$$

2. SPIN-UP SOURCES. DISTINCTION FROM QUADRUPOLAR SIGNALS

The simplest model of a spin-up source is a rigid sphere of radius r_0 spun-up at $t = 0$ to angular momentum **J** by an external torque $\propto 1(t)$. Assuming the center fixed at $\mathbf{x} = 0$, solving Eq. (3) yields (Tolstoy, 2001)

$$\zeta = (8\pi)^{-1}\mu^* r^{-3}(\mathbf{x} \times \mathbf{J})[1(t - r/c + r_0/c) + rc^{-1}\delta(t - r/c + r_0/c)] \qquad (6)$$

For $t > (r - r_0)/c$ this recovers the textbook solution (Weinberg, 1972) for the steadily rotating sphere:

$$\zeta = (8\pi)^{-1}\mu^* r^{-3}(\mathbf{x} \times \mathbf{J}) \qquad (7)$$

The second term in brackets in Eq. (6) is the propagating disturbance generated by the dipolar accelerations of an impulsive spin-up torque. If this external torque is not instantaneous, we replace $1(t)$ by $w(t)$ and $\delta(t)$ by $w'(t)$. The mean strain being $|h| = c^{-1}|\zeta|$ (Tolstoy, 2001), the dominant (second) term gives in the equatorial plane normal to **J**, for large r:

$$h = (8\pi)^{-1}\mu^* r^{-1}c^{-2}Jw'(t - r/c + r_0/c) \qquad (8)$$

This is the shear strain in a plane normal to both the wavefront and **J**.

Equation (8) allows order of magnitude estimates, for example, for a star colliding off-center with a larger one. Assuming a neutron star of mass 2×10^{30} kg, traveling at a relative velocity of 10^6 ms^{-1}, off-center by 10^{11} m, transfering angular momentum $J = 2 \times 10^{47}$ mksu, with $w(t) = tT^{-1}1(t)$ for $t \leq T$ and $=1$ for $t > T$ and $T = 10^3$ s, yields $h \approx 10^{-19}$ for $r \approx 1$ kpc (Tolstoy, 2001). Projected sensitivities of the LIGO system are in the 10^{-21}–10^{-24} range (Abramovici *et al.*, 1992), suggesting that large scale astrophysical phenomena of this type could generate observable GM/GE signals out to ranges $\approx 10^2$ Mpc.

High energy astrophysical events are of course much more complex than this simple spin-up model. We may for instance envision a turbulent cloud of matter, originally accelerated by nongravitational forces, for example, by a supernova explosion. It seems likely that, in such a scenario, classic quadrupole (h_{ij}) gravitational modes would be generated alongside the (essentially dipole) GM/GE waves. It would be difficult and certainly premature here to design realistic models for estimating the relative importance of the GM/GE and quadrupole modes (Tolstoy, 2001). However, in the weak field notation, the GM/GE and quadrupole modes are given respectively by (Adler and Silbergleit, 2000)

$$\Box^2 h_{j0} = -i\mu^*\rho(v_j/c) \qquad (9)$$

$$\Box^2 h_{jk} = -\mu^*\rho(v_j v_k/c^2) \qquad (10)$$

and it is seen that quadrupole source fields are $0(v^2/c^2)$, whereas the GM/GE (dipole) source terms are $0(v/c)$. For a given externally (nongravitationally) generated $\mathbf{v}(t)$ source field, the strains associated with GM/GE modes will therefore have the advantage of a factor of order c/v.

In the present, essentially linear, approximation these two distinct modes of propagation are uncoupled but travel with the same velocity c. It is important to be able to separate and distinguish them in practice.

LIGO detectors were originally designed to observe quadrupole radiation by measuring relative displacements of pairs of points along two perpendicular directions, that is, the extensions and contractions of line segments *in the plane of the wavefront*. This may be illustrated by the deformation of a geometrical figure in this plane (Fig. 1).

GM/GE waves, on the other hand, do not generate relative displacements in the plane of the wavefront but create shear strains in planes *perpendicular to it*, where they will deform a figure such as a square by shearing it, for example, changing it to a lozenge, lengthening one diagonal while shortening the other—effects measurable on a LIGO system. To distinguish GM/GE from quadrupole modes, additional information may be required, such as direction of arrival and/or planes of polarization. Figure 2 shows how, under special conditions, an incident GM/GE wave may give observed extensional strains that, in the absence of knowledge

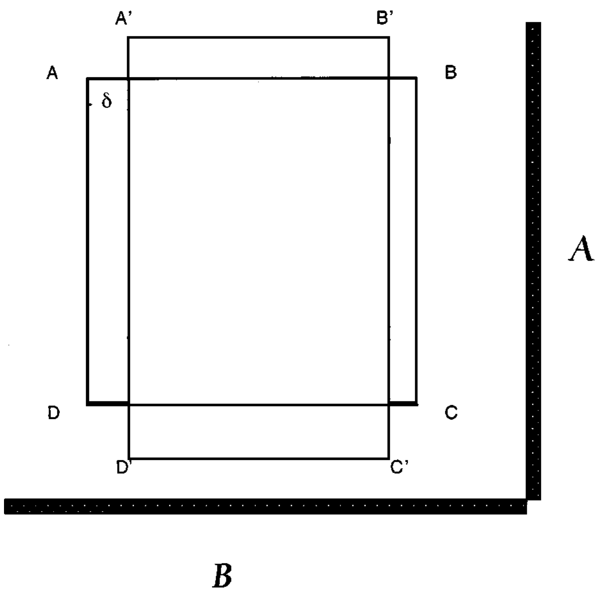

Fig. 1. Quadrupole gravitational wave (h_{ij}) incident normally to the plane of the LIGO array (arms ***A***, ***B***). The phase shown here gives an extension δ of the AD, BC sides of a square (***A*** arm) and a contraction $-\delta$ of the AB, CD sides (***B*** arm).

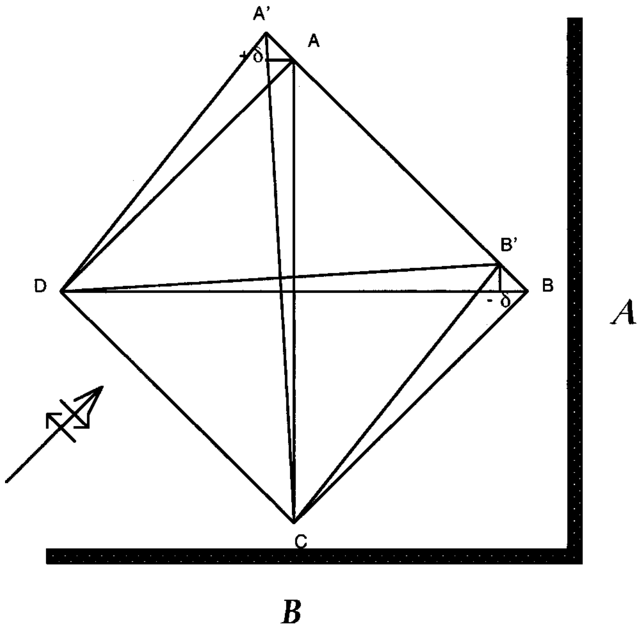

Fig. 2. GM, GE waves (h_{0j}) arriving from the left at a 45° angle with respect to the LIGO arms. The wavefront is normal to the plane of the array. The h_{0j} strain is a shear, that is, a relative displacement of the AB side of a square in the LIGO plane, which we take as $AA' = 2^{1/2}\delta$. To within terms of $0(\delta^2)$, the AC diagonal is stretched by the amount δ (arm ***A***) while DB is shortened by $-\delta$ (arm ***B***). An observer needs additional information (e.g., wavefront's direction) to distinguish this case from that of Fig. 1.

about directions of arrival, are indistinguishable from those of a quadrupolar plane wave.

When GM/GE and quadrupole radiation arrive from the same direction, their contributions may be separated by linear procedures. Refering again to Fig. 2, assume a quadrupolar gravity wavefront coinciding with the GM/GE front. In this case, if Δ is the corresponding length change of a line segment parallel to the LIGO plane, the observed elongations along the array arms are $2^{-1/2}\Delta \pm \delta$ (Fig. 3). The GM/GE and quadrupolar contributions can therefore be separated by adding and subtracting the observed length changes along ***A*** and ***B***. In the three-dimensional case (e.g., using additional, sufficiently distant arrays) more general linear procedures will allow one to separate these modes.

In view of the c/v factor noted above, it is seen that for given observed strains h_{j0} the energy requirements for the source fields of quadrupole gravitational and GM/GE (dipolar) waves will be substantially different—the latter being smaller by

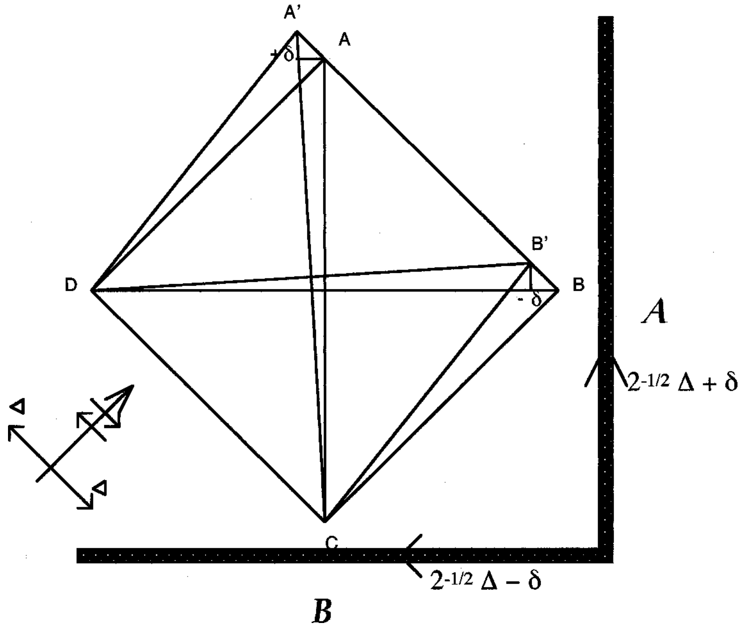

Fig. 3. Quadrupole gravitation (extension Δ of CD, AB segments) and GM/GE waves (extensions $\pm\,\delta$ of the diagonals AC, BD) incident from the same direction as in Fig. 2, combine their strains to yield extensions $2^{-1/2}\Delta \pm \delta$ along LIGO array arms ***A***, ***B***. The mode contributions may thus be separated by adding and subtracting measurements made by the two arms.

a factor of $0(v^2/c^2)$. This fact may be pertinent to the analysis of some of Weber's still controversial results (Weber, 1970), which, when interpreted as quadrupolar waves, appeared to imply unrealistically large rates of energy radiation from the galactic core (Weinberg, 1972).

3. LINEAR ACCELERATION SOURCES

Using currents defined by $\mathbf{j}^* = -\rho\mathbf{v}$ and $\boldsymbol{j}^* = (-ic\rho, -\rho v^k)$ and limiting oneself to em models pertaining to single charge signs (no dielectric effects), Eqs. (3) and (4) are isomorphic to those of the electromagnetic field, and it is possible to transcribe many results of em theory to GM/GE fields.

Thus for a linearly accelerated point mass (or small rigid sphere of mass m) travelling in the direction $\theta = 0$ in a spherical coordinate system, the classic em result for the farfield radiation from an electron in a linear antenna yields, for nonrelativistic velocities

$$\mathbf{E}^* = (4\pi)^{-1}\mu^* r^{-1} m\mathbf{v}_{,t}(t - r/c)\sin\theta \tag{11}$$

The Coulomb terms in $\mathbf{E}^* = -\zeta_{,t} - \nabla\phi^*$ do not contribute to the GM/GE fields (Tolstoy, 2001) and integration gives

$$\zeta = -(4\pi)^{-1}\mu^* r^{-1} m[\mathbf{v}(t - r/c) - \mathbf{v}_0] \sin\theta \tag{12}$$

where $\mathbf{v}_0$ is the steady velocity prior to acceleration by an external force $\mathbf{f}$ applied at $t = 0$, $r = 0$. If we assume this to be impulsive, that is

$$\mathbf{f} = \mathbf{A}\delta(t) \tag{13}$$

we will have

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{A}m^{-1}1(t - r/c) \tag{14}$$

Equation (12) then gives a cylindrically symmetric dipole (shear) strain field with a maximum amplitude at $\theta = \pi/2$:

$$h = (4\pi)^{-1}\mu^* r^{-1} c^{-1}\mathbf{A} = (4\pi)^{-1}\mu^* r^{-1} c^{-1} m(\mathbf{v} - \mathbf{v}_0) \tag{15}$$

A 2×10^{30} kg supernova remnant expelled at 10^6 ms^{-1} yields $h \approx 6.5 \times 10^{-22} \approx 10^{-21}$ at $r = 1$ pc. At 1 kpc this model runs up against LIGO's sensitivity limit of $\approx 10^{-24}$.

When a body (e.g., a supernova remnant) is accelerated, a reacting distribution of mass carries off an equal quantity of momentum of opposite sign, thus creating another GM/GE field that interferes with that of Eq. (15). However, the geometry of the two masses will in general be different and the fields will not cancel (cancellation will only happen in very special cases, for example, in the collision of two identical point masses). While realistic models would thus need to take into account both fields, Eq. (15) still provides an order of magnitude estimate of the generated amplitudes.

One must assume that other high energy events (matter jets? white holes?) are capable of generating GM/GE signals. However, discussion of such models would at this stage be speculative and premature.

4. CONCLUSIONS

Plausible scenarios for GM/GE radiation (shear strains of the metric) generated by transient accelerations of large astronomical bodies emphasize the need for a reliable detection method. LIGO arrays appear to have the necessary sensitivity. But since these wavefields travel with the speed of light c, a complicating factor is the possible simultaneous arrival of quadrupolar gravitational radiation generated in the same region or by the same events. However, knowledge of arrival directions and/or polarization planes allows one to make an unambiguous distinction (Figs. 1–3). This can be secured by the judicious use of LIGO-type arrays, the latest, forthcoming generation of which may make it possible to detect GM/GE events out to source ranges of 1–10^2 Mpc.

It is both possible and essential to distinguish the GM/GE and quadrupole contributions to signals observed on a LIGO array since any significant astrophysical and/or cosmological conclusions to be drawn from such observations will depend strongly upon the type of field being observed.

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